Homework 4 (Due 2/19/2014)

Math 622

February 14, 2014

Fixed small typo in problem 6

1. Let τ and ρ be stopping times with respect to a filtration $\{\mathcal{F}(t);\, t \geq 0\}$.

a) Show that $\tau \wedge \rho\, (= \min\{\tau, \rho\})$ is a stopping time.

b) Show that $\tau \vee \rho\, (= \max\{\tau, \rho\})$ is a stopping time.

2. Let $X = \{X(t);\, t \geq 0\}$ be a stochastic process whose sample paths are all continuous, and assume $X(0)(\omega) = 0$ for all ω. Let $\{\mathcal{F}^X(t)\}_{t\geq 0}$ be the filtration generated by X.

In each case below, determine whether the random time must necessarily be an $\{\mathcal{F}^X(t);\, t \geq 0\}$-stopping time. Justify your answer briefly in each case; you may use the informal rule of thumb and/or general results about stopping times, as in the Lecture Notes for lecture 4.

(a) $T_1 = \inf\{t;\, X^2(t) \geq 1\}$;

(b) $T_2 = \inf\{t;\, \int_0^t X^2(s)\, ds > 1\}$.

(c) $T_3 = \sup\{t;\, t \leq 1 \text{ and } X(t) = 0\}$;

(d) Suppose that $|X(t)|(\omega) > 0$ for all $\omega \in \Omega$ and all $t > 0$, and reconsider $T_2 = \inf\{t;\, \int_0^t X^2(s)\, ds > 1\}$.

(e) $T_5 = \inf\{t; X(t) \geq X(t+1)\}$.

3. Let τ be a stopping time with respect to a filtration, $\{\mathcal{F}(t);\, t \geq 0\}$.

a) Let n be any positive integer. Define a discrete approximation $\tau^{(n)}$ to τ by setting $\tau^{(n)}(\omega) = \frac{k}{n}$ if $\frac{k-1}{n} < \tau \leq \frac{k}{n}$. This approximates τ from above. Show that $\tau^{(n)}$ is an $\{\mathcal{F}(t);\, t \geq 0\}$-stopping time.

b) Let n be any positive integer and define a discrete approximation to τ from below by $\tau_n(\omega) = \frac{k-1}{n}$ if $\frac{k-1}{n} < \tau \leq \frac{k}{n}$. Is τ_n in general an $\{\mathcal{F}(t);\, t \geq 0\}$-stopping time? Explain.

4. (Optional Stopping) Let $\{X_n\}$ be a martingale with respect to the filtration $\{\mathcal{F}_n\}$; thus (i) X_n is $\mathcal{F}_n$-measurable for each n, (ii) $E[|X_n|] < \infty$ for each n, and (iii) $E[X_{n+1} \big| \mathcal{F}_n] = X_n$ for each n. Let τ be a stopping time with respect to $\{\mathcal{F}_n\}$. Show that the stopped process $X_{n\wedge\tau}$ is also a martingale with respect to $\{\mathcal{F}_n\}$.

Hint: Write $X_{n\wedge\tau} = \sum_{k=0}^{n} X_k \mathbf{1}_{\{\tau=k\}} + X_n \mathbf{1}_{\{\tau>n\}}$. Observe that $\{\tau > n\}$ is $\mathcal{F}_n$-measurable (why?).

5. (Extra Credit: 5pts) Let W be a Brownian motion and let $\{\mathcal{F}(t);\, t \geq 0\}$ be filtration for W. We claimed in class that if $Y(t) = \int_0^t \alpha(s)\, dW(s)$, and if τ is a stopping time with respect to $\{\mathcal{F}(t);\, t \geq 0\}$, then $Y(t \wedge \tau) = \displaystyle\int_0^t \mathbf{1}_{[0,\tau)}(s)\alpha(s)\, dW(s)$.

In this problem we want to show a special case of this. Assume that $\tau(\omega) \leq T$ for all ω where T is positive constant. We want to show

$$W(\tau) = \int_0^T \mathbf{1}_{[0,\tau)}(s)\, dW(s) \tag{1}$$

a) Case (i): The stopping time τ takes values in a discrete set $t_0 = 0 < t_1 < t_2 < \cdots < t_n = T$. In this case, identify random variables $\alpha_0, \alpha_1, \ldots, \alpha_{n-1}$ such that α_k is $\mathcal{F}(t_k)$-measurable for each k, and

$$\mathbf{1}_{[0,\tau)}(s) = \sum_{k=0}^{n-1} \alpha_k \mathbf{1}_{[t_k,t_{k+1})}(s).$$

This shows that $\mathbf{1}_{[0,\tau)}(s)$ is a simple process as defined in Shreve, section 4.2.1. (What we call α_k here is what is denoted by $\triangle(t_k)$ in §4.2.1.) Now apply the definition of the stochastic integral for simple processes in equation (4.2.2) to prove (1).

b) Case (ii). The general case. Let τ be any stopping time with $\tau(\omega) \leq T$ for all ω. Let $\tau^{(n)}$ be the approximation to τ constructed in part (a) of exercise 3. Since $\tau^{(n)}$ takes values in a discrete set, equation (1) is true when τ is replace by $\tau^{(n)}$, for each n. Argue that $\lim_{n\to\infty} \tau_n(\omega) = \tau(\omega)$ for all ω, and conclude that (1) is true for τ.

6. Consider the two asset, risk-neutral model

$$\begin{aligned} dS_1(t) &= rS_1(t)\, dt + \sigma_1(S_1(t), S_2(t))S_1(t)\, d\widetilde{W}_1(t) \\ dS_2(t) &= rS_2(t)\, dt + \sigma_2(S_1(t), S_2(t))S_2(t)\, d\widetilde{W}_2(t) \end{aligned}$$

where $\widetilde{W}_1$ and $\widetilde{W}_2$ are independent Brownian motions and $\sigma_1(x_1, x_2)$ and $\sigma_2(x_1, x_2)$ are strictly positive, bounded, differentiable functions. You may take as known that, given $S_1(0)$ and $S_2(0)$, this system has a unique solution which is a Markov process. Let τ be the first time that $S_1(t)$ hits the level $B > 0$, and let ρ be the first time $S_2(t)$ hits B. Consider an option which knocks out if either $S_1(t)$ hits B or $S_2(t)$ hits B, and otherwise pays $(S_1(T)S_2(T) - K)^+$ at time T. Denote its price by $V(t)$.

a) Show that $V(t) = \mathbf{1}_{\{\tau\wedge\rho>t\}} v(t, S_1(t), S_2(t))$, where $v(t, x_1, x_2) = 0$ if $x_1 \geq B$ or $x_2 \geq B$, and otherwise,

$$v(t, x_1, x_2) = e^{-r(T-t)} \tilde{E}\Big[\quad \mathbf{1}_{\max_{[t,T]} S_1(u) < B} \mathbf{1}_{\max_{[t,T]} S_2(u) < B} \Big(S_1(T) S_2(T) - K \Big)^+ \Big| S_1(t) = x_1, S_2(t) = x_2 \Big].$$

b) Show that $e^{-r(t\wedge\tau\wedge\rho)} v(t, S_1(t\wedge\tau\wedge\rho), S_2(t\wedge\tau\wedge\rho))$ is a martingale and derive a partial differential equation for $v(t, x_1, x_2)$. Specify the domain in (x_1, x_2)-space on which this equation is valid and all boundary and terminal conditions.